MODULAR FORMS 2019: THETA FUNCTIONS WEEKS OF MAY 5,12 2019

ZEÉV RUDNICK

Contents

1. Quadratic forms and lattices

Consider an quadratic form $A[x] = \sum a_{ij}x_ix_j$ in r variables. We assume that $A = A^T \gg 0$ is positive definite. Then we can write $A = B^TB$ with B invertible.

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We will say that A is <u>integral</u> if

$$a_{ii} \in \mathbb{Z} \text{ and } 2a_{ij} \in \mathbb{Z} \quad i \neq j$$

Therefore if A is integral, we have $A[x] \in \mathbb{Z}$ whenever $x \in \mathbb{Z}^r$

The problems we shall want to address

a) Given $n \in \mathbb{Z}_{\geq 0}$, when can we solve $A[x] = n$ for $x \in \mathbb{Z}^r$? For instance, which integers are sums of two squares, or three squares, or four squares...

b) Denote $r_A(n) = \#\{x \in \mathbb{Z}^r : A[x] = n\}$. We want a formula, or at least an approximation, for $r_A(n)$.

1.1. **Terminology: quadratic forms and lattices.** There are two ways to speak about (positive definite) quadratic forms:

a) Start with $\mathbb{R}^r$ equipped with the standard inner product $\langle x, y\rangle = \sum_{j=1}^r x_j y_j$. We denote $||x||^2 = \langle x, x\rangle$. Take a lattice $L \subset \mathbb{R}^r$, and a basis $L = \oplus_{j=1}^r \mathbb{Z}w_j$. The associated *Gram matrix* is

$$A = (a_{ij}) = (\langle \omega_i, \omega_j\rangle)$$

which is clearly symmetric, and is positive definite, since

$$A[x] := x^T A x = \sum_{ij} a_{ij} x_i x_j = ||\sum x_j w_j||^2$$

is non-negative, and because $\{w_j\}$ are linearly independent, can only vanish if $x = \vec{0}$. We say that the quadratic form A is induced on the lattice L.

b) Fix the integer lattice $\mathbb{Z}^r$. Given a symmetric, positive definite matrix $A = A^T \gg 0$ define an inner product by

$$\langle x, y\rangle_A := x^T A y$$

Then we can obtain a lattice $L = L_A \subset \mathbb{R}^r$ and a basis w_j of L so that the corresponding Gram matrix (w.r.t. the <u>standard</u> inner product) is A: $a_{ij} = \langle w_i, w_j\rangle$.

Indeed, find an orthonormal basis $\{u_j\}$ of $\mathbb{R}^r$ w.r.t. the inner product $\langle \bullet, \bullet\rangle_A$: $\langle u_i, u_j\rangle_A = \delta_{ij}$. Define a linear map $\Omega \in \mathrm{GL}(r, \mathbb{R})$ by

$$\Omega u_j = e_j$$

where $e_j = (0, \dots, 0, 1, 0 \dots)$ is the standard basis of $\mathbb{R}^r$. Then

$$A = \Omega^T \cdot \Omega,$$

because

$$u_i^T \Omega^T \Omega u_j = e_i^T e_j = \delta_{ij} = \langle u_i, u_j\rangle_A = u_i^T A u_j$$

Now take

$$L := \Omega \mathbb{Z}^r, \quad w_j = \Omega e_j$$

Then

$$\langle w_i, w_j \rangle = \langle \Omega e_i, \Omega e_j \rangle = e_i^T \Omega^T \Omega e_j = e_i^T A e_j a_{ij}$$

so that the Gram matrix of L w.r.t. the basis $\{w_j\}$ is precisely A.

1.2. **Volume and determinant of a lattice.** Let $L \subset \mathbb{R}^r$ be a lattice, and $A = A^T \gg 0$ an associated Gram matrix. Define the determinant of the lattice to be $\det L := \det A$.

Choosing a different basis $\{w'_j\}$ gives a Gram matrix A', related to A via

$$A' = C^T A C$$

where $C = (c_{k\ell})$ is the change of basis matrix, given by

$$w'_i = \sum_k c_{ki} w_i$$

The matrix C is an integral matrix with integral inverse: $C \in \mathrm{GL}(r, \mathbb{Z})$. In particular $\det C \in \mathbb{Z}^* = \{\pm 1\}$. Therefore $\det A' = \det A$, so that $\det L := \det A$ is well-defined.

Alternatively, recall that if $\Omega : \mathbb{R}^r \to \mathbb{R}^r$ is the linear map such that for an orthonormal basis u_j for the inner product $\langle x, y \rangle_A = x^T A y$, we have $u_j = \Omega e_j$ with e_j the standard basis, then $A = \Omega^T \Omega$ and then

$$\det L = \det A = (\det \Omega)^2$$

We can also describe $\det L$ in terms of the volume of any fundamental domain D for L:

$$\det L = (\operatorname{vol} D)^2 = \Big(\operatorname{vol}(\mathbb{R}^r / L) \Big)^2$$

Indeed, since $A = A^T \gg 0$, we can write $A = \Omega^T \cdot \Omega$, and if $\Omega = (w_1 | \dots | w_r)$ then $L = \Omega \mathbb{Z}^r = \oplus \mathbb{Z} w_j$ (now think of w_j as column vectors). Then $D = \Omega \cdot [0,1]^r$ is a fundamental domain for L, and therefore its volume is $\operatorname{vol} D = \det \Omega = \sqrt{\det A}$.

Lemma 1.1. *If $L' \subset L$ is a lattice of index $[L : L']$ then*

$$\det L' = [L : L']^2 \cdot \det L$$

Proof. Let $\{\ell_j : j = 1, \dots, d\} \subset L$ be a set of representatives for L/L'. Then as a fundamental domain for L' we may take

$$D' = \cup_{j=1}^d (\ell_j + D)$$

and then clearly $\operatorname{vol}(D') = d \operatorname{vol}(D)$, so that

$$\det L' = (\operatorname{vol} D')^2 = d^2 (\operatorname{vol} D)^2 = d^2 \det L$$

$\square$

1.3. **The dual lattice.** If $L \subset \mathbb{Z}^r$ is a lattice, define
$$L^* = \{x \in \mathbb{R}^r : \langle x, \ell\rangle \in \mathbb{Z}, \quad \forall \ell \in L\}$$
This turns out to also be a lattice, called the dual lattice to L.

Example: If $L = \oplus \mathbb{Z} m_j e_j$ (e_j is the standard basis of $\mathbb{Z}^r$) then $L^* = \oplus \mathbb{Z}\frac{1}{m_j} e_j$.

The dual lattice of $L = \Omega \mathbb{Z}^r$ is then
$$L^* = (\Omega^T)^{-1}\mathbb{Z}^r$$
Indeed.
$$x \in L^* \Leftrightarrow x^T \cdot \Omega n \in \mathbb{Z}, \forall n \in \mathbb{Z}^r \Leftrightarrow \Omega^T x \in (\mathbb{Z}^r)^* = \mathbb{Z}^r \Leftrightarrow x \in (\Omega^T)^{-1}\mathbb{Z}^r$$

Consequently, we find
$$\det L^* = (\det(\Omega^T)^{-1})^2 = (\det \Omega)^{-2} = (\det L)^{-1}$$
so that
$$\det(L^*) = \frac{1}{\det L}$$
We say a lattice L is self-dual if $L^* = L$. Necessarily, we have $\det L = 1$ in this case.

1.4. **Direct sums.** Given two quadratic forms, we can create a new one, called the direct sum, as follows: If $L \subset \mathbb{R}^r$, and $L' \subset \mathbb{R}^s$ are lattices, then we create a lattice $L \oplus L' \subset \mathbb{R}^r \times \mathbb{R}^s \simeq \mathbb{R}^{r+s}$. Choosing bases $\{w_i\}$ of L and $\{w'_j\}$ of L' gives Gram matrices A and A', and the Gram matrix associated to the basis $\{(w_i, w_j) : 1 \le i \le r, 1 \le j \le s\}$ is the direct sum $\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}$. Clearly $\det(L \oplus L') = \det L \cdot \det L'$.

1.5. **Integrality.** We say that the lattice L (or the quadratic form A) is *integral* if $x^T A y \in \mathbb{Z}$ for all $x, y \in \mathbb{Z}^r$, equivalently if $\langle \ell, \ell'\rangle \in \mathbb{Z}$ for all $\ell, \ell' \in L$. For instance, if $L \subseteq \mathbb{Z}^r$ then L is integral.

For an integral lattice, we have by definition $L \subseteq L^*$.

We say that L is *even* if $\langle \ell, \ell\rangle \in 2\mathbb{Z}$ is even for all $\ell \in L$. Necessarily this implies that L is integral, since
$$\langle \ell, \ell'\rangle = \frac{1}{2}\Big(|\ell + \ell'|^2 - |\ell|^2 - |\ell'|^2\Big)$$
Exercise 1. Show that if L is integral then
$$\det L = \#(L^*/L) = [L^* : L]$$

2. Theta functions

2.1. **Definition of theta functions.** Let $L \subset \mathbb{R}^r$ be a lattice. We define the associated theta function $\theta_L(\tau)$, for $\tau \in \mathbb{H}$ by

$$\theta_L(\tau) := \sum_{\ell \in L} q^{\langle \ell, \ell \rangle / 2}, \qquad q = e^{2\pi i \tau}$$

Note that

$$\theta_L(\tau) = \sum_{n \geq 0} r_L(n) q^{n/2}$$

where

$$r_L(n) = \#\{\ell \in L : \langle \ell, \ell \rangle = n\} = \#\{x \in \mathbb{Z}^r : \frac{1}{2} Q[x] = n\}$$

in particular, $r_L(0) = 1$. If L is integral then the sum is over $n \in \mathbb{N}$.

The sum is absolutely convergent for all $\tau \in \mathbb{H}$ (equivalently, $|q| < 1$) because as we saw, $r_L(n)$ grows at most polynomially. Hence $\theta_L(\tau)$ is analytic in $\mathbb{H}$. If L is integral then it also satisfies $\theta_L(\tau + 2) = \theta_L(\tau)$. If moreover L is **even** then $\theta_L(\tau + 1) = \theta_L(\tau)$.

For instance, taking $L = \mathbb{Z}^k$ the standard lattice, we have

$$\theta_{\mathbb{Z}^k}(\tau) = \sum_{n \geq 0} r_k(m) q^{m/2}$$

with

$$r_k(m) = \#\{x \in \mathbb{Z}^k : m = x_1^2 + \cdots + x_k^2\}$$

2.2. **The Fourier transform and the Poisson summation formula.**

Definition. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consisting of smooth functions f so that f and all its derivatives decay rapidly:

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}^d, \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty\}$$

where

$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \quad \partial^\beta f := \frac{\partial^{\beta_1 + \cdots + \beta_d} f}{\partial^{\beta_1} x_1 \ldots \partial^{\beta_d} x_d}.$$

Clearly $C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \geq 1$.

Example: The Gaussian e^{-x^2} lies in $\mathcal{S}(\mathbb{R})$.

The Fourier transform of a Schwarz function $f \in \mathcal{S}$ is defined as

$$\mathcal{F}(f) = \widehat{f}(y) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot y} dx$$

Using integration by parts, it is easy to see that if $f \in \mathcal{S}$ then so is $\widehat{f}$.

Theorem 2.1. *Let $g(x) = e^{-\pi x^2}$. Then $\widehat{g} = g$.*

It is easy to check that the Fourier transform intertwines dilation operators: If $\lambda > 0$, and $(D_\lambda f)(x) := f(x/\lambda)$ ($f \in \mathcal{S}$), then

$$\widehat{(D_\lambda f)}(y) = \lambda^d \widehat{f}(\lambda y)$$

The Poisson summation formula

Theorem 2.2. *Let $L \subset \mathbb{R}^r$ be a lattice. Then for $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\sum_{\ell \in L} f(\ell) = \frac{1}{\sqrt{\det L}} \sum_{\lambda \in L^*} \widehat{f}(\lambda)$$

In particular,

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{m \in \mathbb{Z}^d} \widehat{f}(m)$$

2.3. **The functional equation of theta functions.**

Theorem 2.3. *Let $L \subset \mathbb{R}^r$ be a lattice. Then*

$$\theta_L(\tau) = \frac{1}{\sqrt{\det L}} (-i\tau)^{-r/2} \theta_{L^*}(-\frac{1}{\tau})$$

Proof. We do it in the special case of the standard one-dimensional lattice $L = \mathbb{Z} \subset \mathbb{R}$ so that

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2}$$

and we claim

$$\theta(-\frac{1}{\tau}) = \sqrt{-i\tau}\theta(\tau)$$

where the branch of the square root is determined so that $\sqrt{w} > 0$ if $w > 0$.

Indeed, since both sides are analytic in $\mathbb{H}$, it suffices to show that they coincide on the imaginary axis $\tau = iy$, $y > 0$, that is to show

$$\theta(i/y) = y^{1/2}\theta(iy)$$

But

$$\theta(iy) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = \sum_{n \in \mathbb{Z}} f_y(n)$$

where $f_y = D_{1/\sqrt{y}} f_1$, $f_1(x) = e^{-\pi x^2}$, whose Fourier transform is

$$\widehat{f_y}(x) = \widehat{D_{1/\sqrt{y}} f_1}(x) = \frac{1}{\sqrt{y}} \widehat{f_1}(\frac{x}{\sqrt{y}}) = \frac{1}{\sqrt{y}} e^{-\pi x^2/y}$$

Applying the Poisson summation formula we have

$$\theta(y) = \sum_{n\in\mathbb{Z}} f_y(n) = \sum_{m\in\mathbb{Z}} \widehat{f}_y(m) = \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{y}} e^{-\pi m^2/y} = \frac{1}{\sqrt{y}}\theta(\frac{i}{y})$$

as claimed. □

2.4. **The theta function of even self-dual lattices.** Assume now that $L \subset \mathbb{R}^r$ is even and self-dual. (It turns out, and we will see this below, that such lattices only exist if $8 \mid r$). Then the theta function satisfies

$$\theta_L(\tau+1) = \theta(\tau), \quad \theta(-1/\tau) = (-i\tau)^{r/2}\theta(\tau)$$

and if $8 \mid r$ then θ_L is a weakly modular form of weight $r/2$ for $\mathrm{SL}(2,\mathbb{Z})$. Since

$$\theta(i\infty) = \#\{\ell \in L : \langle \ell, \ell \rangle = 0\} = 1$$

for any lattice, we have the condition of being "bounded at infinity", and we obtain

Theorem 2.4. *Let L be an even, self-dual lattice of dimension r. Then $\theta_L \in M_{r/2}(\mathrm{SL}(2,\mathbb{Z}))$ is a modular form of weight $r/2$ for $\mathrm{SL}(2,\mathbb{Z})$.*

Let L be an even self-dual lattice. Then we saw that $\theta_L \in M_k(\mathrm{SL}(2,\mathbb{Z}))$, $k = \dim L/2$. Thus we can write $\theta_L = cE_k + f$ for a cusp for $f \in S_k$ and a scalar $c \in \mathbb{C}$, where $E_k = 1 + \gamma_k \sum_{n\geq 1} \sigma_{k-1}(n) q^n$ is the normalized Eisenstein series. Since both $E_k(i\infty) = 1 = \theta_L(i\infty)$, we must have $c = 1$, so that

$$\theta_L = E_{\dim L/2} + f$$

Using Hecke's bound on the size of Fourier coefficients of cusp forms, we obtain

Corollary 2.5. *Let L be an even self-dual lattice of dimension $\dim L = 8s$. Then for all $n \geq 1$*

$$r_L(2n) = \gamma_{4s}\sigma_{4s-1}(n) + O_L(n^{2s})$$

Recall that

$$\gamma_4 = 240, \quad \gamma_8 = 480, \quad \gamma_{12} = \frac{65520}{691}, \quad \gamma_{16} = \frac{16320}{3617}.$$

Since $\sigma_{4s-1}(n) \approx n^{4s-1}$, the remainder term is of smaller order. In particular, we find that for $n \gg_L 1$, $r_L(2n) \neq 0$, so that there is a vector in L of norm $2n$.

3. The lattices $E(r)$

3.1. **Construction.** We now construct some important examples of integral (and even) lattices: Let

$$F = \{x \in \mathbb{Z}^r : \langle x, x\rangle = 0 \bmod 2 \leftrightarrow \sum_{j=1}^{r} x_j = 0 \bmod 2\} \subset \mathbb{Z}^r$$

This is sublattice of index two in $\mathbb{Z}^r$, since F is the kernel of the surjective map $\mathbb{Z}^r \to \mathbb{Z}/2\mathbb{Z}$, $x \mapsto \sum_j x_j \bmod 2$. Let

$$\delta := (\frac{1}{2}, \frac{1}{2}, \dots \frac{1}{2}) \in \mathbb{R}^r$$

Now note that if r is *even*, then $2\delta \in F$. From now on assume then that r is even. Let

$$E(r) = F + \mathbb{Z}\delta$$

be the lattice generated by F and the vector δ. Since $2\delta \in F$ but $\delta \notin F$, we have

$$[E(r) : F] = 2$$

ie F is a sublattice of index two in $E(r)$. Since also $[\mathbb{Z}^r : F] = 2$, we must have

$$\operatorname{vol} E(r) = \operatorname{vol} \mathbb{Z}^r = 1$$

Because $\delta \notin \mathbb{Z}^r$, also $E(r) \not\subseteq \mathbb{Z}^r$.

Lemma 3.1. *a) If $4 \mid r$ then $E(r)$ is an integral lattice*
b) If $8 \mid r$ then $E(r)$ is even.

Proof. a) Write $\ell = f + m\delta \in L$, $\ell' = f' + m'\delta \in L$, with $f, f' \in F$, $m, m' \in \mathbb{Z}$. Then

$$\langle \ell, \ell'\rangle = \langle f, f'\rangle + m\langle \delta, f'\rangle + m'\langle f', \delta\rangle + mm'\langle \delta, \delta\rangle$$

Now $\langle f, f'\rangle \in \mathbb{Z}$ since $F \subset \mathbb{Z}^r$, and $\langle f, \delta\rangle = \frac{1}{2}\sum_j f_j \in \mathbb{Z}$ since $\sum f_j \in 2\mathbb{Z}$ by definition of F. So it suffices to check when $\langle \delta, \delta\rangle$ is an integer. But

$$\langle \delta, \delta\rangle = \frac{r}{4} \in \mathbb{Z} \Leftrightarrow r = 0 \bmod 4$$

so that $E(r)$ is an integral lattice if (and only if) $4 \mid r$.

b) To check when is $E(r)$ even, we use the same approach: If $f = (f_1, \dots, f_r) \in F$ then

$$\langle \ell, \ell\rangle = \langle f, f\rangle + 2m\langle f, \delta\rangle + m^2\langle \delta, \delta\rangle \equiv m\sum_j f_j + m^2\frac{r}{4} \bmod 2$$

since $\langle f, f\rangle \in 2\mathbb{Z}$. Now $m^2 = m \bmod 2$, and $\sum f_j \equiv \langle f, f\rangle = 0 \bmod 2$ so that

$$\langle \ell, \ell\rangle \equiv m\frac{r}{4} \bmod 2$$

This will be even for all $m \in \mathbb{Z}$ iff $r/4 \in 2\mathbb{Z}$, that is iff $8 \mid r$. $\square$

Lemma 3.2. *If $4 \mid r$ then $E(r)^* = E(r)$, that is $E(r)$ is self-dual.*

Proof. We have

$$\mathbb{Z}^r \supset F_r \subset E(r)$$

both with index 2. Taking duals reverses inclusions and preserves indices, and we get

$$\mathbb{Z}^r = (\mathbb{Z}^r)^* \subset F_r^* \supset E(r)^*$$

with both inclusions of index 2. Hence

$$2^2 \det(E(r)^*) = \det((F_r)^*) = 2^2 \det((\mathbb{Z}^r)^*) = 2^2 \det \mathbb{Z}^r = 2^2 \cdot 1$$

so that

$$\det(E(r)^*) = 1$$

Also, since $4 \mid r$, $E(r)$ is integral so that $E(r) \subseteq E(r)^*$, and hence

$$1 = \det(E(r)^*) = [E(r)^* : E(r)]^2 \det E(r) = [E(r)^* : E(r)]^2 \cdot 1$$

so that $[E(r)^* : E(r)] = 1$. Hence $E(r)^* = E(r)$ so that $E(r)$ is self-dual if $4 \mid r$. $\square$

Corollary 3.3. *$E(8k)$ is an even, self-dual lattice.*

We can create more even self-dual lattices by taking direct sums (which preserve these properties). Thus in dimension 16 we have the lattice $E(8) \oplus E(8)$, as well as $E(16)$. We will soon see that these are not isomorphic.

3.2. **The lattice $E(8)$.** As an example, take the case of dimension 8, where we have the $E(8)$ lattice. Then there are no cusp forms of weight 4, so that

$$\theta_L = E_4$$

Hence

$$\sum_{n\geq 0} r_L(2n)q^n = 1 + 240 \sum_{n\geq 1} \sigma_3(n)q^n$$

so that

$$r_L(2n) = 240\sigma_3(n)$$

in particular $r_L(1) = 240$, so that there are 240 vectors of norm 2.

It turns out (e.g. by using the (Smith-Minkowski-) Siegel mass formula) that there is only one even self-dual lattice in dimension 8, namely $E(8)$ (proved by Mordell).

A basic question about any lattice is the existence of "short" vectors. For even lattices, the shortest possible is vectors of norm $|\ell|^2 = 2$.

Consider the case of the lattice $E(8)$. As we saw by identifying the theta function of $E(8)$, there are exactly 240 vectors of norm 2. The following vectors have norm 2 (where e_i are the standard basis vectors):

$$\pm e_i \pm e_k, \quad 1 \leq i < k \leq 8,$$

$$\frac{1}{2}\sum_{j=1}^{8} \epsilon_j e_j, \quad \epsilon_j = \pm 1, \quad \prod_{j=1}^{8} \epsilon_j = 1$$

this is a set of 240 vectors of norm 2, hence we have identified all minimal vectors in $E(8)$.

Exercise 2. Check that these vectors all lie in $E(8)$.

Exercise 3. As a basis for $E(8)$ we can take the following subset of minimal vectors

$$v_1 = \frac{1}{2}(e_1+e_8) - \frac{1}{2}(e_2+\cdots+e_7),\ v_2 = e_1+e_2,\ v_{i+1} = e_i - e_{i-1}, 2 \leq i \leq 7$$

The corresponding Gram matrix is (Figure 1)

$$\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}$$

FIGURE 1. A Gram matrix for $E(8)$.

3.3. Dimension 16: non-isomorphic lattices with the same theta function.

It turns out (E. Witt, using the "mass formula") that there are only 2 even self-dual lattices in dimension 16, namely $E(8) \oplus E(8)$ and $E(16)$.

Proposition 3.4. *For any even self-dual lattice L in dimension 16 we have*

$$\theta_L = E_8$$

(E_8 denotes the normalized Eisenstein series).

Indeed, $\theta_L \in M_8(\mathrm{SL}(2,\mathbb{Z})) = \mathbb{C}E_8$ and since M_8 is one-dimensional, θ_L has to be a multiple of E_8. They have the same zero-th fourier coefficient since $\theta_L(i\infty) = 1$ for any L, and likewise $E_8(i\infty) = 1$. so that

$$\theta_L = E_8 = 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n$$

so the number of vectors of norm 2 is given by

$$r_L(2) = 480\sigma_7(1) = 480$$

that is, any self-dual even lattice in dimension 16 has exactly 480 vectors of norm 2.

As a corollary, we see that the vectors of norm 2 in $E(16)$ are precisely the 480 vectors $\pm e_i \pm e_k$, $1 \leq i < k \leq 16$. Their span is contained in $F_{16} \subset \mathbb{Z}^{16}$ so they do not span $E(16)$.

Exercise 4. For any $m \geq 2$, the vectors of norm 2 in $E(8m)$ are precisely $\pm e_i \pm e_k$, $1 \leq i < k \leq 8m$ (there are $2 \cdot 8m \cdot (8m-1)$ such vectors).

Note that these vectors lie in $F_{8m} \subset \mathbb{Z}^{8m}$ so they do not span $E(8m)$.

Corollary 3.5. *$E(16)$ and $E(8) \oplus E(8)$ are not isomorphic.*

Indeed, the vectors of norm 2 span $E(8)$, hence also $E(8) \oplus E(8)$, but they do not span $E(16)$, because in that case they all lie in $F_{16} \subset \mathbb{Z}_{16}$.

3.4. **Isospectral tori.** Note that we saw

$$\theta_{E(16)} = \theta_{E(8)\oplus E(8)} = E_8$$

Thus we have found a pair of lattice with the same theta function which are not isomorphic.

This was used by Milnor to observe that the corresponding flat tori $\mathbb{R}^{16}/L$ ($L = E(8) \oplus E(8)$ or $E(16)$) are isospectral but not isometric. So one cannot "hear the shape of a drum" in dimension 16...

Indeed, for any lattice $L \subset \mathbb{R}^r$, we define a flat torus $\mathbb{R}^r/L$, that is the metric is the flat Euclidean metric, and functions on $\mathbb{R}^r/L$ are just functions on $\mathbb{R}^r$ which are periodic with respect to L: $f(x+\ell) = f(x)$, $\forall \ell \in L$.

An important example are the following: Given a vector $\ell^* \in L^*$ in the dual lattice, let

$$f(x) = f_{\ell^*}(x) = e^{2\pi i \langle x, \ell^* \rangle}$$

Then check that $f(x+\ell) = f(x)$, $\forall \ell \in L$ (since $\langle \ell, \ell^* \rangle \in \mathbb{Z}$). It is a fact that these give an orthonormal basis of $L^2(\mathbb{R}^r/L)$.

Now note that these are eigenfunctions of the Euclidean Laplacian $\Delta = \sum_{j=1}^r \frac{\partial^2}{\partial x_j^2}$, with eigenvalue

$$-\Delta f = 4\pi^2 |\ell^*|^2 f$$

and consequently the spectrum of the Laplacian on this flat torus consists of the numbers $4\pi^2 n$, where $n = |\ell^*|^2$ is a norm of a dual lattice vector, with multiplicities $r_{L^*}(n)$.

Thus the tori associated to $L = E(8) \oplus E(8)$ or $E(16)$ are isospectral. It also turns out that $\mathbb{R}^r/L$ is isometric to $\mathbb{R}^r/L'$ (as Riemannian manifolds) if and only if L and L' are isometric (as lattices). Thus the two flat tori $\mathbb{R}^{16}/L$ ($L = E(8) \oplus E(8)$ or $E(16)$) are isospectral but not isometric.

3.5. **Dimension 24.** Here we must have $\theta_L = E_{12} + f$, with $f \in S_{12} = \mathbb{C}\Delta$. Thus

$$\theta_L = E_{12} + c_L \Delta$$

We have

$$E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n$$

and comparing the coefficient of q we obtain (note $\sigma_{11}(1) = 1 = \tau(1)$)

$$r_L(2) = \frac{65520}{691}\sigma_{11}(1) + c_L \tau(1) = \frac{65520}{691} + c_L$$

so that

$$c_L = r_L(2) - \frac{65520}{691}$$

so that θ_L is determined by $r_L(2)$, the number of vectors of norm 2.

Therefore, for all $n \geq 1$ we obtain

$$r_L(2n) = \frac{65520}{691}\sigma_{11}(n) + \Big(r_L(2) - \frac{65520}{691}\Big)\tau(n)$$

Exercise 5. Ramanujan's congruence: Noting that 691 is a prime,

$$\tau(n) = \sigma_{11}(n) \bmod 691$$

It turns out (H. Niemeier 1968) that there are exactly 24 inequivalent even self-dual lattices of dimension 24. Only one of them, the Leech lattice, does not contain a vector of norm 2, so that for it

$$r_{\mathrm{Leech}}(2n) = \frac{65520}{691}\Big(\sigma_{11}(n) - \tau(n)\Big)$$

In dimension 32, there are $>$ 80 million even self-dual lattices.

4. $E(8)$ AND SPHERE PACKINGS

Given a discrete set of points $X \subset \mathbb{R}^d$, such that $||x-y|| \geq 2$ for all distinct $x, y \in X$ (this is a normalization), the associated *sphere packing* is

$$\mathcal{P} := \cup_{x \in X} B_d(x,1)$$

where $B_d(x,1) \subset \mathbb{R}^d$ is the ball of radius 1 around x. If X is a lattice, then $\mathcal{P}$ is called a *lattice packing*. The question is to find "dense" sphere packings, that is those for which the "fraction" of space covered by the balls of $\mathcal{P}$ is as large as possible. To quantify this, one defines the *density* of the packing as

$$\delta(\mathcal{P}) := \limsup_{r \to \infty} \frac{\operatorname{vol}\Big(B(0,r) \cap \mathcal{P}\Big)}{\operatorname{vol} B(0,r)}$$

The d-dimensional sphere packing constant is the maximal density in dimension d

$$\delta_d = \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \delta(\mathcal{P})$$

In dimension 1 we clearly have $\delta_1 = 1$. In dimension 2, it was long known (A. Thue 1910, L. Fejes Tóth 1943) that the maximal density is achieved by the hexagonal lattice packing (Figure 2), for which $\delta_2 = \pi/\sqrt{12} \approx 0.90690\ldots$, where each disk touches six others.

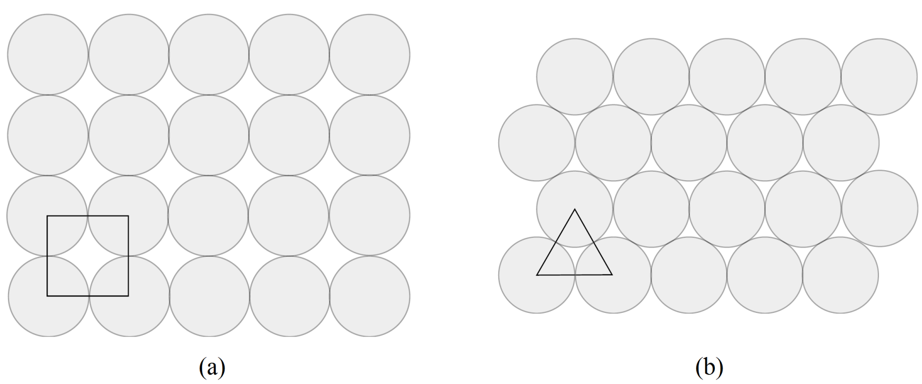

FIGURE 2. The hexagonal circle packing in the plane (b). The packing (a) has density $\pi/4 = 0.7853\ldots$ and the packing (b) has density $\pi/\sqrt{12} \approx 0.90690\ldots$ (source: http://blog.kleinproject.org/?p=742)

Exercise 6. Justify the densities of the planar packings of Figure 2.

In dimension 3, Johanes Kepler (1611) conjectured that no arrangement of equally sized spheres filling space has density greater than $\pi/\sqrt{18} \approx 0.74048\ldots$. This density is attained by the face-centered cubic packing (the familiar pyramidal piling of oranges seen in grocery stores, Figure 3) and also by uncountably many nonlattice packings, all obtained as follows: Start with a layer of spheres in a hexagonal lattice, then put the next layer of spheres in the lowest points you can find above the first layer, and so on. At each step there are two choices of where to put the next layer, so this natural method of stacking the spheres creates an uncountably infinite number of equally dense packings, the best known of which are called cubic close packing and hexagonal close packing. Gauss (1831) proved that this was the best lattice packing, but for many years it was not proved that it was the best sphere packing. A machine aided proof was announced in 1998 by T. Hales, with a formal machine verifiable proof published in 2015.

FIGURE 3. Packing oranges source: http://blog.kleinproject.org/?p=742

In 2016, Maryna Viazovska proved that in dimension 8, the optimal sphere packing is the lattice packing for the scaled lattice $\frac{1}{\sqrt{2}}E(8)$, with density $\pi^4/384 \approx 0.25367\ldots$, and a few days later, together with H. Cohn, A. Kumar and S.D. Miller, showed that in dimension 24, the optimal packing is the lattice packing for the Leech lattice. Both these answers were conjectured for some time. These are the only dimensions where the optimal packing is expected to be a lattice packing, and in no other dimension is the answer known.

Appendix A. Nonexistence of even self dual lattices

We use the information on the theta function to deduce that there are no even self-dual lattices in dimensions which are not a multiple of 8.

Lemma A.1. *If L is an even self-dual lattice then $8 \mid \dim L$.*

Proof. Assume that $8 \nmid \dim L = r$. Then either $4r = 4 \bmod 8$ (if r is odd) or $2r = 4 \bmod 8$ (if $r = 2 \bmod 4$) or $r = 4 \bmod 4$. Then replacing L by either $L \oplus L \oplus L \oplus L$ or $L \oplus L$, we arrive to the situation $\dim L = 4 \bmod 8$. Then the functional equation reads

$$\theta_L(-1/\tau) = (-i\tau)^{\dim L/2}\theta_L(\tau) = -\tau^{\dim L/2}\theta_L(\tau)$$

since $\dim L = 4 \bmod 8$.

Also $\theta_L(\tau+1) = \theta_L(\tau)$.

We define a right action of $\mathrm{SL}(2,\mathbb{Z})$ on functions by the "slash operator"

$$(f|^k_\gamma)(\tau) := j(\gamma,\tau)^{-k} f(\gamma(\tau))$$

where $j(\gamma,\tau) = c\tau + d)$ for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. Note that it is indeed a right action:

$$\left(f|^k_\gamma\right)|^k_\delta = f|^k_{\gamma\delta}$$

since by the chain rule $j(\gamma\delta,\tau) = j(\gamma,\delta\tau)j(\delta,\tau)$ so that

$$\begin{aligned}\left(f|^k_\gamma\right)|^k_\delta(\tau) = j(\delta,\tau)^{-k} f|^k_\gamma(\delta(\tau)) &= j(\delta,\tau)^{-k} j(\gamma,\delta\tau)^{-k} f(\gamma(\delta(\tau))) \\ &= j(\gamma\delta,\tau)^{-k} f((\gamma\delta)(\tau)) = f|^k_{\gamma\delta}(\tau)\end{aligned}$$

Note that $f \in M_k$ means $f|^k_\gamma = f$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Then the functional equation for θ_L reads

$$\theta_L|S = -\theta_L$$

Now apply the slash operator $|^k_R$ with the element $R := ST$ and $k = \dim L/2$:

$$\theta_L|_R = \theta_L|_{ST} = (\theta_L|_S)|_T = -\theta_L|_T = -\theta_L$$

On the other hand, since $R^3 = I$ we have

$$\theta_L = \theta_L|_{R^3} = ((\theta_L|_R)|R)|_R = -\theta_L$$

which forces $\theta_L = 0$. But $\theta_L(i\infty) = 1$, giving a contradiction. $\square$